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Best L_p Approximation with Multiple Constraints for $1 \leqslant p < \infty$

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The problem considered in this paper is best L_p approximation with multiple constraints for $1 \leqslant p < \infty$. Characterizations of best L_p approximations from multiple n-convex splines and functions are established and the relationship between them is investigated. Applications to best monotone convex approximation are studied. © 1991 Academic Press, Inc.

1. INTRODUCTION

In this paper, we consider best L_p approximation with multiple constraints for $1 \leqslant p < \infty$. The classes of approximating functions are the class of multiple n-convex splines and the class of multiple n-convex functions, which are defined below.

A real-valued function g is said to be *n-convex* in (0, 1) if for any $n+1$ distinct points $x_0, x_1, ..., x_n$ in (0, 1), the nth order divided difference is nonnegative, i.e.,

$$[x_0, x_1, ..., x_n]\, g \geqslant 0.$$

The set of n-convex functions is a convex cone. Note that 1-convex functions are nondecreasing and 2-convex functions are convex in the usual sense.

It is known (e.g., [2]) that if g is an n-convex function on (0, 1) then $g^{(n-2)}$ exists and is convex on (0, 1). Hence, $g^{(n-2)}$ is absolutely continuous on any closed subinterval of (0, 1), the $(n-1)$st left-derivative $g_-^{(n-1)}$ exists and is left-continuous and nondecreasing in (0, 1), the $(n-1)$st right-derivative $g_+^{(n-1)}$ exists and is right-continuous and nondecreasing in (0, 1), $g^{(n-1)}$ exists a.e. in (0, 1), and $g^{(n-1)} = g_-^{(n-1)} = g_+^{(n-1)}$ a.e. in (0, 1). If $g \in C^n[0, 1]$, then g is n-convex if and only if $g^{(n)} \geqslant 0$. The set of n-convex

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functions contains the subspace of polynomials of degree $n-1$. Some additional properties of n-convex functions can be found in [2, 11, 16, 19].

Given $0 \leqslant m \leqslant n$, g is said to be *(m, n)-convex* if $(-1)^i g$ is $(m+i)$-convex for $i=0, 1, ..., n-m$. Note that for $n > m$, (m, n)-convex functions are functions with multiple constraints. Let $K_{m,n}$ denote the set of (m, n)-convex functions. Then clearly $K_{m,n}$ is the finite intersection of k-convexity cones. The finite intersections of generalized convexity cones with respect to an ECT-system were defined in [20, 21]. Clearly, $K_{m,n}$ is a finite intersection of the convexity cone with respect to the ECT-system $\{1, x, x^2, ..., x^{n-1}\}$.

From the above definition, (n, n)-convex functions are n-convex functions and $(0, n)$-convex functions are n-time monotone functions. For some applications of n-time monotone functions, see [18] and other references therein. In addition, $(0, \infty)$-convex functions are completely monotone functions (see [17]). More generally, we define $(m, n)_\sigma$-convexity. Let $\sigma = (\sigma_0, \sigma_1, ..., \sigma_{n-m})$, where each σ_i is 1 or -1. A function g is said to be *$(m, n)_\sigma$-convex* if $\sigma_i(-1)^i g$ is $(m+i)$-convex, for $i=0, 1, ..., n-m$. In this paper, for the sake of simplicity we restrict ourselves to (m, n)-convex functions. All results we obtain here can be extended to the setting with arbitrary σ without any difficulty.

Let $K^p_{m,n}$ denote the intersection of $K_{m,n}$ and $L_p = L_p[0, 1]$. Then $K^p_{m,n}$ is a closed convex cone in L_p. Given a partition Δ of $[0, 1]$, with $\Delta: 0 = x_0 < x_1 < \cdots < x_{k+1} = 1$, let $S^k_n(\Delta)$ denote the space of polynomial splines of degree $n-1$ with k simple knots at $x_1, ..., x_k$, i.e.,

$$S^k_n(\Delta) = \operatorname{span}\{(1-x)^{i-1}, i = 1, 2, ..., n, (x_j - x)^{n-1}_+, j = 1, 2, ..., k\}.$$

Define

$$S^k_{m,n}(\Delta) = S^k_n(\Delta) \cap K_{m,n}. \tag{1.1}$$

Since polynomials of degree $n-1$ are contained in both $S^k_n(\Delta)$ and $K_{m,n}$, $S^k_{m,n}(\Delta)$ is a nonempty convex cone. In particular, $S^0_{m,n}(\Delta)$ is the set of (m, n)-convex polynomials of degree $n-1$.

Given $f \in L_p[0, 1]$, $s^* \in K^p_{m,n}$ (resp., $S^k_{m,n}(\Delta)$) is called a *best (m, n)-convex* (resp., *(m, n)-convex spline*) *L_p approximation* to f if

$$\|f - s^*\|_p = \inf\{\|f - s\|_p : s \in K^p_{m,n} (\text{resp., } S^k_{m,n}(\Delta))\}. \tag{1.2}$$

The existence of a best n-convex L_1 approximation was proved in [7] and [16] independently, and uniqueness is proved under some additional restrictions in [22]. The characterizations of best 1-convex (nondecreasing) L_p approximations for $1 \leqslant p < \infty$ were established in [12, 13]. A partial characterization of a best n-convex L_1 approximation was proved in [22]. The complete characterization of a best n-convex L_p approximation, for $1 \leqslant p < \infty$, is considered in [14, 15, 19]. Existence of a best n-convex uniform approximation was proved in [3, 24]. Burchard [4] and Brown [1] have characterized best uniform n-convex approximation. Some additional properties of best uniform n-convex approximation are considered in [23].

For $1 \leqslant p < \infty$, the existence of a best approximation to $f \in L_p[0, 1]$ from $S^k_{m,n}(\Delta)$ follows from the fact that $S^k_{m,n}(\Delta)$ is a finite dimensional, closed subset of L_p. For $1 < p < \infty$, unicity follows from the fact that L_p is strictly convex. For $p = 1$, unicity was proved by Pence in [9]. In Section 2, the characterizations of best (m, n)-convex spline L_p approximations for $1 \leqslant p < \infty$ are established. As consequences, we also consider best L_p approximation by n-convex splines of degree $n-1$.

For $1 < p < \infty$, the existence of a unique best L_p approximation from $K^p_{m,n}$ follows from the facts that $L^p_{m,n}$ is closed and convex in the reflexive Banach space L_p and that the L_p norm is uniformly convex. In Section 3, we prove the existence of a best L_1 approroaximation of $f \in L_1[0, 1]$ from $K^1_{m,n}$ and characterize best L_p approximation to a function f in $L_p[0, 1]$ from $K^p_{m,n}$ for $1 \leqslant p < \infty$. An interesting relationship between best L_p approximations to $f \in C[0, 1]$ from $S^k_{m,n}(\Delta)$ and $K^p_{m,n}$ is investigated in Section 4. In Section 5, best monotone convex L_p approximations are studied and best convex L_p approximation is characterized in terms of best monotone convex L_p approximations.

2. Best L_p Approximation from $S^k_{m,n}(\Delta)$

By a corollary of the Hahn–Banach Theorem (see [5, 6]), if K_p is a convex cone in $L_p[0, 1]$ for $1 \leqslant p < \infty$, then

(i) for $1 < p < \infty$, $s^*_p \in K_p$ is a best L_p approximation to $f \in L_p[0, 1]$ from K_p if and only if

$$\int_0^1 s^*_p \phi_p = 0, \tag{2.1}$$

and

$$\int_0^1 s\phi_p \leqslant 0, \qquad \text{for all} \quad s \in K_p, \tag{2.2}$$

where $\phi_p = \operatorname{sign}(f - s^*_p)\, |f - s^*_p|^{p-1}$; and

(ii) for $p = 1$, $s^*_1 \in K_1$ is a best L_1 approximation to $f \in L_1[0, 1]$ from K_1 if and only if there exists a $\phi_1 \in L_\infty$ with $\|\phi_1\|_\infty = 1$ and $\int_0^1 \phi_1(f - s^*_1) = \|f - s^*_1\|_1$, satisfying (2.1) and (2.2) with $p = 1$.

The above result shall be referred to as the duality theorem.

Let $d\mu$ be a signed measure of bounded variation on (0, 1). The dual cone to a cone K of functions is the set of signed measures $d\mu$ such that

$$\int_0^1 g(x)\,d\mu(x) \geqslant 0 \qquad \text{for all} \quad g \in K.$$

With this definition, the above duality theorem can be restated as follows: For $1 < p < \infty$ $s_p^* \in K_p$ is a best L_p approximation to $f \in L_p$ from K_p if and only if ϕ_p is orthogonal to s_p^* and $-\phi_p(x)\,dx$ is in the dual cone to K_p. For $p = 1$, we can similarly restate the duality theorem. The dual cone to a finite intersection of generalized convexity cones with respect to an ECT-system was characterized by Ziegler in [20, 21].

By applying the duality theorem, we have the following characterization of best L_p approximation to $f \in L_p$ from $S_{m,n}^k(\Delta)$ for $1 \leqslant p < \infty$. Let $N_{m,n} = \{m+1, ..., n\}$ and $N_n = N_{0,n}$.

THEOREM 2.1 (Characterization). *For* $1 \leqslant p < \infty$, *let* $f \in L_p[0, 1]$ *and let* $s_p^* \in S_{m,n}^k(\Delta)$.

(a) *For* $1 < p < \infty$, *let* $\phi_p = \operatorname{sign}(f - s_p^*)\,|f - s_p^*|^{p-1}$, *and*

$$H_{p,i}(x) = \{1/(i-1)!\} \int_0^x (x-t)^{i-1} \phi_p(t)\,dt, \qquad x \in [0, 1],\ i \in N_n. \quad (2.3)$$

Then s_p^* *is the best* L_p *approximation to* f *from* $S_{m,n}^k(\Delta)$ *if and only if*

(i) $H_{p,i}(1) = 0$, $i \in N_m$;

(ii) $(-1)^m H_{p,i}(1) \leqslant 0$, $i \in N_{m,n}$;

(iii) $(-1)^m H_{p,n}(x_j) \leqslant 0$, $j \in N_k$;

(iv) *if* $(-1)^m H_{p,i}(1) < 0$ *for some* $i \in N_{m,n}$, *then* $s_p^{*(i-1)}(1) = 0$;

(v) *if* $(-1)^m H_{p,n}(x_j) < 0$ *for some* $j \in N_k$, *then* $s_p^{*(n-1)}(x_j^-) = s_p^{*(n-1)}(x_j^+)$.

(b) *For* $p = 1$, s_1^* *is a best* L_1 *approximation from* $S_{m,n}^{k,1}(\Delta)$ *to* f *if and only if there exists a* $\phi_1 \in L_\infty$ *with* $\|\phi_1\|_\infty = 1$ *and* $\int_0^1 \phi_1(f - s_1^*) = \|f - s_1^*\|_1$ *satisfying* (i)–(v) *of part* (a) *with* $p = 1$. *We call* ϕ_1 *an associated functional of* s_1^*.

Proof. (a) This proof will depend on the above duality theorem. Since $S_{m,n}^k(\Delta)$ is a closed convex cone in $L_p[0, 1]$, by the duality, s_p^* is the best approximation to f from $S_{m,n}^k(\Delta)$ if and only if

$$\int_0^1 s_p^* \phi_p = 0, \quad (2.4)$$

and

$$\int_0^1 s\phi_p \leqslant 0 \qquad \text{for all} \quad s \in S^k_{m,n}(\Delta). \tag{2.5}$$

(Necessity) First, note that $(1-x)^{i-1}/(i-1)!$, $-(1-x)^{i-1}/(i-1)! \in S^k_{m,n}(\Delta)$ for $i = 1, 2, ..., m$. By substituting these functions into inequality (2.5), we find

$$\int_0^1 \{(1-x)^{i-1}/(i-1)!\}\, \phi_p(x)\, dx = 0, \qquad i = 1, 2, ..., m.$$

This proves (i).

Next, since $(-1)^m(1-x)^{i-1}/(i-1)! \in S^k_{m,n}(\Delta)$, $i = m+1, ..., n$, by using (2.5) once again, we obtain (ii). Similarly, in (2.5), let $s = (-1)^m(x_j - x)^{n-1}_+/(n-1)!$, $j = 1, 2, ..., k$, and we have

$$\int_0^{x_j} \{(-1)^m(x_j - x)^{n-1}/(n-1)!\}\, \phi_p(x)\, dx \leqslant 0, \qquad j = 1, 2, ..., k$$

Now, by integrating by parts and by using (i),

$$\begin{aligned}\int_0^1 s_p^*(x)\, \phi_p(x)\, dx &= \int_0^1 (-1)^m H_{p,m}(x)\, s_p^{*(m)}(x)\, dx \\ &= \sum_{i=m}^{n-1} (-1)^i H_{p,i+1}(1)\, s_p^{*(i)}(1) \\ &\quad + \sum_{j=1}^{k} (-1)^n H_{p,n}[x_j)[s_p^{*(n-1)}(x_j^+) - s_p^{*(n-1)}(x_j^-)],\end{aligned}$$

where the last equality holds because s_p^* is a polynomial of degree $n-1$ on each subinterval (x_j, x_{j+1}). Combining the above equation with (2.4) gives

$$\sum_{i=m}^{n-1} (-1)^i H_{p,i+1}(1)\, s_p^{*(i)}(1) + \sum_{j=1}^{k} (-1)^n H_{p,n}(x_j)[s_p^{*(n-1)}(x_j^+) - s_p^{*(n-1)}(x_j^-)] = 0. \tag{2.6}$$

Since $s_p^* \in K_{m,n}$, $(-1)^{i-m} s_p^{*(i)}(1) \geqslant 0$ and $(-1)^{n-m}[s_p^{*(n-1)}(x_j^+) - s_p^{*(n-1)}(x_j^-)] \geqslant 0$. It follows from (ii) and (iii) that each term in (2.6) is nonpositive. Hence,

$$(-1)^m H_{p,i+1}(1)\, s_p^{*(i)}(1) = 0, \qquad i = m, m+1, ..., n-1, \tag{2.7}$$

and

$$(-1)^m H_{p,n}(x_j)[s_p^{*(n-1)}(x_j^+) - s_p^{*(n-1)}(x_j^-)] = 0, \qquad j = 1, 2, ..., k. \quad (2.8)$$

Then (2.7) implies (iv) and (2.8) implies (v).

(Sufficiency) If $s_p^* \in S_{m,n}^k(\Delta)$ satisfying conditions (i)–(v), then by integation by parts, it is easy to verify that (2.4) and (2.5) hold. Therefore, s_p^* is the best approximation to f from $S_{m,n}^k(\Delta)$.

(b) The proof is similar to (a). Thus, we omit the details. This proves Theorem 2.1.

We remark that since $H'_{p,i}(x) = H'_{p,i-1}(x)$, conditions (i)–(v) of Theorem 2.1 can be restated in terms of $H_{p,n}$ and its derivatives. For example, conditions (i) and (ii) are equivalent to $H_{p,n}^{(n-i)}(1) = 0$, $i \in N_m$, and $(-1)^m H_{p,n}^{(n-i)}(1) \leqslant 0$, $i \in N_{m,n}$, respectively.

In order to derive some structural properties of a best approximation, we introduce some additional notation. Let $1 \leqslant p < \infty$ and $\phi_p \in (L_p)^*$, the dual space of L_p. Define $H_{p,i}$ as in (2.3),

$$I(\phi_p) = \{i \in N_{m,n} : (-1)^m H_{p,i}(1) < 0\}, \qquad (2.9)$$

and

$$J(\phi_p) = \{j \in N_k : (-1)^m H_{p,n}(x_j) < 0\}. \qquad (2.10)$$

We define a subspace of $S_n^k(\Delta)$ by

$$S_n^{k*}(\Delta, \phi_p) = \{s \in S_n^k(\Delta) : s^{(i)}(1) = 0, \ i \in I(\phi_p);$$
$$s^{(n-1)}(x_j^-) = s^{(n-1)}(x_j^+), \ j \in J(\phi_p)\}. \qquad (2.11)$$

It is easily proved that $S_n^{k*}(\Delta, \phi_p)$ has a basis

$$\{(1-x)^{i-1}, \ i \in N_n - I(\phi_p), \ (x_j - x)_+^{n-1}, \ j \in N_k - J(\phi_p)\}. \qquad (2.12)$$

The next theorem gives an alternate characterization of best L_p approximation from $S_{m,n}^k(\Delta)$, which indicates that best L_p approximation from the convex set $S_{m,n}^k(\Delta)$ is equivalent to best L_p approximation from the subspace $S_n^{k*}(\Delta, \phi_p)$.

THEOREM 2.2. *Let $1 \leqslant p < \infty$ and let $f \in L_p[0, 1]$.*

(a) *For $1 < p < \infty$, $s_p^* \in S_{m,n}^k(\Delta)$ is the best L_p approximation to f from $S_{m,n}^k(\Delta)$ if and only if s_p^* is the best L_p approximation from $S_n^{k*}(\Delta, \phi_p)$, where*

$$\phi_p = \text{sign}(f - s_p^*) \, |f - s_p^*|^{p-1}.$$

(b) *For* $p=1$, $s_1^* \in S_{m,n}^k(\Delta)$ *is a best* L_1 *approximation to* f *from* $S_{m,n}^k(\Delta)$ *if and only if there is a* $\phi_1 \in L_\infty$ *with* $\|\phi_1\|_\infty = 1$ *and* $\int_0^1 \phi_1(f - s_1^*) = \|f - s_1^*\|_1$, *such that* s_1^* *is a best* L_1 *approximation to* f *from* $S_n^{k*}(\Delta, \phi_1)$.

Proof. Let $1 < p < \infty$. By Theorem 2.1, s_p^* is the best L_p approximation to f from $S_{m,n}^k(\Delta)$ if and only if conditions (i)–(v) are satisfied. It follows from the definitions of $I(\phi_p)$ and $J(\phi_p)$ that conditions (i)–(v) are equivalent to the conditions

$$\int_0^1 (1-t)^{i-1} \phi_p(t)\, dt = 0, \qquad i \in N_n - I(\phi_p), \tag{2.13}$$

and

$$\int_0^1 (x_j - t)_+^{n-1} \phi_p(t)\, dt = 0, \qquad j \in N_k - J(\phi_p). \tag{2.14}$$

This is equivalent to the statement that s_p^* is the best L_p approximation to f from $S_n^{k*}(\Delta, \phi_p)$, since $S_n^{k*}(\Delta, \phi_p)$ is a finite dimensional subspace of $S_n^k(\Delta)$ and $S_n^{k*}(\Delta, \phi_p)$ has the basis (2.12).

The proof of (b) is similar to that of (a). This completes the proof.

In the rest of this section we apply the general results that we just obtained to best L_p approximations from $S_{n,n}^k(\Delta)$, the set of n-convex splines of degree $n-1$, for $1 \leqslant p < \infty$.

COROLLARY 2.1. *For* $1 \leqslant p < \infty$, *let* $f \in L_p[0, 1]$ *and let* $s_p^* \in S_{n,n}^k(\Delta)$.

(a) *For* $1 < p < \infty$, *the following statements are equivalent*:

(1) s_p^* *is the best* L_p *approximation to* f *from* $S_{m,n}^k(\Delta)$;

(2) s_p^* *satisfies three conditions*

(i) $H_{p,i}(1) = 0$, $i = 1, 2, ..., n$,

(ii) $(-1)^n H_{p,n}(x_j) \leqslant 0$, $j = 1, 2, ..., k$,

(iii) *if* $(-1)^n H_{p,n}(x_j) < 0$ *for some* $j \in N_k$, *then* $s_p^{*(n-1)}(x_j^-) = s_p^{*(n-1)}(x_j^+)$.

(3) s_p^* *is the best* L_p *approximation to* f *from* $S_n^{k**}(\Delta, \phi_p)$ *defined by*

$$S_n^{k**}(\Delta, \phi_p) = \{s \in S_n^k(\Delta) : s^{(n-1)}(x_j^-) = s^{(n-1)}(x_j^+),\ j \in J(\phi_p)\},$$

where $J(\phi_p) = \{j \in N_k : (-1)^n H_{p,n}(x_j) < 0\}$.

(b) *For* $p=1$, s_1^* *is a best* L_1 *approximation to* f *from* $S_{n,n}^k(\Delta)$ *if and only if there exists a* $\phi_1 \in L_\infty$ *with* $\|\phi_1\|_\infty = 1$ *and* $\int_0^1 \phi_1(f-s_1^*) = \|f-s_1^*\|_1$, *satisfying the conditions* (i)–(iii) *of part* (a) *with* $p=1$, *and if and only if there exists a* ϕ_1 *as above such that* s_1^* *is the best* L_p *approximation to* f *from* $S_n^{k**}(\Delta, \phi_1)$.

3. Best L_p Approximation from $K_{m,n}^p$

In this section, we consider best L_p approximation to $f \in L_p$ from $K_{m,n}^p$ for $1 \leqslant p < \infty$.

First of all, we study the existence of a best (m, n)-convex L_1 approximation. It will be proved to be a consequence of an existence theorem in a recent paper [16] by Ubhaya. We first state a definition and a theorem that appear in [16]. Let H be the set of all extended real-valued function on [0, 1]. We say $P \subset H$ is sequentially closed if it is closed under pointwise convergence of sequences of functions. We denote by $\bar{P}$ the smallest superset of P which is sequentially closed.

THEOREM 3.1 [16]. *Let* P *be a nonempty set in* H. *Assume the following two conditions are satisfied*:

(1) $P \cap L_p = \bar{P} \cap L_p$;

(2) *There exists a positive integer* z *which depends only upon* P, *and the following holds*: *If* $k \in P$, *there exist an integer* $1 \leqslant r \leqslant z$ *and points* $\{x_i : i = 0, 1, ..., r\}$ *with* $0 = x_0 < x_1 < \cdots < x_r = 1$ *so that* k *is monotone on each interval* (x_{i-1}, x_i).

Then a best approximation to f *in* L_p *from* $P \cap L_p$ *exists for* $1 \leqslant p < \infty$.

The following theorem is a consequence of Theorem 3.1.

THEOREM 3.2. *Let* $f \in L_1[0, 1]$. *Then there exists a best* (m, n)*-convex* L_1 *approximation to* f.

Proof. Let

$$K_i = \{g \in H : (-1)^i g \text{ is } (m+i)\text{-convex}\}.$$

Then $K_{m,n} = \bigcap_{i=0}^{n-m} K_i$. Thus,

$$K_{m,n} \cap L_1 = \left(\bigcap_{i=0}^{n-m} K_i\right) \cap L_1 = \bigcap_{i=0}^{n-m} \{K_i \cap L_1\}.$$

By Proposition 3.4 of [16], we have $K_i \cap L_1 = \overline{K_i} \cap L_1$. Hence,

$$\begin{aligned} K_{m,n} \cap L_1 &= \bigcap_{i=0}^{n-m} (\overline{K_i} \cap L_1) \\ &= \left(\bigcap_{i=0}^{n-m} \overline{K_i} \right) \cap L_1 \\ &= \overline{\left(\bigcap_{i=0}^{n-m} \overline{K_i} \right)} \cap L_1 \\ &= \overline{\left(\bigcap_{i=0}^{n-m} K_i \right)} \cap L_1 \\ &= \overline{K_{m,n}} \cap L_1 . \end{aligned}$$

Therefore, condition (1) in Theorem 3.1 is satisfied. In addition, since an (m, n)-convex function is m-convex, by a property of m-convec functions (see [15, 16]), condition (2) is also satisfied. It follows from Theorem 3.1 that there exists a best L_1 approximation to f from $K^1_{m,n}$. This completes the proof.

Next we establish a characterization of best L_p approximation by (m, n)-convex functions, for $1 \leqslant p < \infty$. To do this, we first prove the following:

LEMMA 3.1. *Let g be (m, n)-convex on $[0, 1]$. Then $g^{(n-1)}(1^-)$ and $g^{(m+i)}(1^-)$, $i = 0, 1, ..., n-m-2$, are finite.*

Proof. Since g is (m, n)-convex and $(-g)$ is $(m+1)$-convex. We then find that $g^{(m)}$ is nonincreasing and $g^{(m)}(x) \geqslant 0$ for all $x \in (0, 1)$. Hence, for an arbitrarily small ε with $0 < \varepsilon < \frac{1}{2}$,

$$0 \leqslant g^{(m)}(1-\varepsilon) \leqslant g^{(m)}(\tfrac{1}{2}).$$

However, $g^{(m)}(\frac{1}{2}) < +\infty$. It follows that $g^{(m)}(1^-)$ is finite. This proof can be completed by induction on i.

We are now ready to state our main theorem in this section.

THEOREM 3.3 (Characterization). *For $1 \leqslant p < \infty$, let $f \in L_p[0, 1]$ and let $g_p^* \in K^p_{m,n}$.*

(a) *For $1 < p < \infty$, let $\phi_p = \operatorname{sign}(f - g_p^*) |f - g_p^*|^{p-1}$, and define $H_{p,i}$ as (2.3). Then g_p^* is the best L_p approximation to f from $K^p_{m,n}$ if and only if*

(i) $H_{p,i}(1) = 0$, $i = 1, 2, ..., m$;

(ii) $(-1)^m H_{p,i}(1) \leqslant 0$, $i = m+1, ..., n$;

(iii) $(-1)^m H_{p,n}(x) \leqslant 0$, $x \in [0, 1]$;

(iv) *if* $(-1)^m H_{p,i}(1) < 0$ *for some* $i \in \{m+1, ..., n\}$, *then* $g_p^{*(i-1)}(1^-) = 0$;

(v) *if* $(-1)^m H_{p,n}(x) < 0$ *for some* $x \in (0, 1)$, *then* g_p^* *is a polynomial of degree* $n-1$ *in a neighborhood of* x.

(b) *For* $p = 1$, g_1^* *is a best* L_1 *approximation to* f *from* $K_{m,n}^p$ *if and only if there exists* $\phi_1 \in L_\infty$ *with* $\|\phi_1\|_\infty = 1$ *and* $\int_0^1 \phi_1(f - g_1^*) = \|f - g_1^*\|_1$, *satisfying the conditions* (i)–(v) *of part* (a) *with* $p = 1$.

Proof. (a) This proof depends on the duality theorem, as the proof of Theorem 23.1.

(Necessity) The proof for (i)–(iii) is similar to the proof for (i)–(iii) in Theorem 2.1. To prove (iv) and (v), we establish the following integration by parts:

$$\int_0^1 g_p^* \phi_p = \sum_{i=m}^{n-1} (-1)^i H_{p,i+1}(1)\, g_p^{*(i)}(1^-) + (-1)^n \int_0^1 H_{p,n}\, d(g_p^{*(n-1)}). \tag{3.1}$$

A similar reasoning as in the proof of Theorem 1 of [15] gives

$$\int_0^1 g_p^* \phi_p = (-1)^m \int_0^1 H_{p,m}\, g_p^{*(m)},$$

and $H_{p,m} g_p^{*(m)} \in L_1[0, 1]$. By Lemma 3.1, $g_p^{*(m)}(1^-)$ is finite, and thus, for an arbitrarily small $\varepsilon > 0$, $H_{p,m+1}\, g_p^{*(m+1)} \in L_1[\varepsilon, 1]$. Hence, integration by parts yields

$$\int_\varepsilon^1 H_{p,m}\, g_p^{*(m)} = H_{p,m+1}(1)\, g_p^{*(m)}(1^-) - H_{p,m+1}(\varepsilon)\, g_p^{*(m)}(\varepsilon) - \int_\varepsilon^1 H_{p,m+1}\, g_p^{*(m+1)}. \tag{3.2}$$

If $g_p^{*(m)}(0^+)$ is finite, then by letting $\varepsilon \to 0$ in (3.2), we obtain

$$\int_0^1 H_{p,m}\, g_p^{*(m)} = H_{p,m+1}(1)\, g_p^{*(m)}(1^-) - \int_0^1 H_{p,m+1}\, g_p^{*(m+1)}, \tag{3.3}$$

and $H_{p,m+1}\, g_p^{*(m+1)} \in L_1[0, 1]$. Otherwise, we must have $|g_p^{*(m)}(0^+)| = +\infty$. Since $g_p^{*(m)}$ is nonincreasing, there exists a $t \in (0, 1)$ such that $|g_p^{*(m)}|$ is nonincreasing on $(0, t)$. Whenever $0 < \varepsilon < t$,

$$|H_{p,m+1}(\varepsilon)\, g_p^{*(m)}(\varepsilon)| \leq |g_p^{*(m)}(\varepsilon)| \leq |g_p^{*(m)}(\varepsilon)| \int_0^\varepsilon |H_{p,m}| \leq \int_0^\varepsilon |g_p^{*(m)} H_{p,m}|.$$

Since $H_{p,m} g_p^{*(m)} \in L_1[0, 1]$, we have $\lim_{\varepsilon \to 0} \int_0^\varepsilon |g_p^{*(m)} H_{p,m}| = 0$. By the above inequality,

$$\lim_{\varepsilon \to 0} \int_0^\varepsilon |g_p^{*(m)}(\varepsilon) H_{p,m+1}(\varepsilon)| = 0.$$

Letting $\varepsilon \to 0$, we also come out with (3.3) and $H_{p,m+1} g_p^{*(m+1)} \in L_1[0, 1]$. This procedure can be repeated to obtain (3.1).

Combining the duality theorem and (3.1) yields

$$\sum_{i=m}^{n-1} (-1)^i H_{p,i+1}(1) g_p^{*(i)}(1^-) + (-1)^n \int_0^1 H_{p,n} d(g_p^{*(n-1)}) = 0.$$

The definition of an (m, n)-convex function together with (ii) and (iii) implies that

$$(-1)^i H_{p,i+1}(1) g_p^{*(i)}(1^-) = 0, \qquad i = m, ..., n-1, \tag{3.4}$$

and

$$\int_0^1 H_{p,n} d(g_p^{*(n-1)}) = 0. \tag{3.5}$$

Equations (3.4) and (3.5) give (iv) and (v), respectively.

(Sufficiency) Assume $g_p^* \in K_{m,n}^p$ and it satisfies conditions (i)–(v). Then by (3.1), (2.1) holds. Also, (3.1) is true if we replace g_p^* by any $g \in K_{m,n}^p$. Hence, (2.2) holds by using conditions (i)–(v). Consequently, g_p^* is a best L_p approximation to f from $K_{m,n}^p$, since $K_{m,n}^p$ is a convex cone.

(b) Since the proof for $p = 1$ is similar, we omit the details. This completes the proof.

This theorem can be extended to characterize a best L_p approximation from $(m, n)_\sigma$-convex functions.

4. A Relationship between Best Approximations from $S_{m,n}^k(\Delta)$ and $K_{m,n}^p$

We assume throughout this section that $f \in C[0, 1]$ and $1 \leqslant p < \infty$. To establish a relationship between best L_p approximations to f from $S_{m,n}^k(\Delta)$ and $K_{m,n}^p$, we need the following theorem.

THEOREM 4.1. *For $1 \leqslant p < \infty$, let $f \in C[0, 1]$ and $g_p^* \in K_{m,n}^p$ be given.*

Assume that $f \neq g_p^*$ *a.e. in* $[0, 1]$ *and that* $f - g_p^*$ *has a finite number of sign changes in* $(0, 1)$. *Let*

$$\phi_p = \begin{cases} \operatorname{sign}(f - g_p^*)|f - g_p^*|^{p-1} & 1 < p < \infty \\ \operatorname{sign}(f - g_p^*) & p = 1 \end{cases}$$

and define $H_{p,i}(x)$ *as* (2.3). *Then* g_p^* *is a best* L_p *approximation to* f *from* $K_{m,n}^p$ *if and only if* (i)–(iv) (*of Theorem* 3.3) *hold with* $1 \leqslant p < \infty$, *and*

(v)′ g_p^* *is a spline of degree* $n-1$ *with simple knots* $\xi_1, \xi_2, ..., \xi_r$, *the distinct zeros of* $H_{p,n}$ *in* $(0, 1)$.

Proof. Let $g_p^* \in K_{m,n}^p$ be a best (m, n)-convex L_p approximation to f. By the hypothesis, $f - g_p^*$ has a finite number of sign changes in $(0, 1)$. Assume that the number of sign changes of $f - g_p^*$ in $(0, 1)$ is N. By the definition of ϕ_p for $1 \leqslant p < \infty$, ϕ_p has N sign changes in $(0, 1)$. Since $\phi_p = H_{p,n}^{(n)}$, by Rolle's Theorem, $H_{p,n}$ has at most $N+n$ zeros in $(0, 1)$, computing multiplicities. Let $\xi_1 < \xi_2 < \cdots < \xi_r$ be the distinct zeros of $H_{p,n}$ in $(0, 1)$, where $r \leqslant N + n$. Let $\xi_0 = 0$ and $\xi_{r+1} = 1$. Note that $(-1)^m H_{p,n}(x) \leqslant 0$, $x \in [0, 1]$. Hence,

$$(-1)^m H_{p,n}(x) < 0 \qquad \text{for} \quad x \in (\xi_i, \xi_{i+1}), \quad i = 0, 1, ..., r.$$

Thus, by (v) of Theorem 3.3, g_p^* is a polynomial of degree $n-1$ on each subinterval (ξ_i, ξ_{i+1}). Since $g_p^* \in C^{n-2}(0, 1)$, g_p^* is a spline of degree $n-1$ with simple knots $\xi_1, \xi_2, ..., \xi_r$.

Conversely, let g_p^* satisfy the assumptions and conditions (i)–(iv) and (v)′. If

$$(-1)^m H_{p,n}(x_0) < 0 \qquad \text{for some} \quad x_0 \in (0, 1),$$

then $x_0 \notin \{\xi_1, \xi_2, ..., \xi_r\}$. Hence, $x_0 \in (\xi_j, \xi_{j+1})$ for some index $j \in \{0, 1, ..., r\}$. By (v)′ g_p^* is a polynomial of degree $\leqslant n-1$ on (ξ_j, ξ_{j+1}), which is a neighborhood of x_0. Thus, by Theorem 3.3, g_p^* is a best L_p-approximation to f from $K_{m,n}^p$. This completes the proof.

By Theorem 2.1, 3.3 and the above theorem, the following theorem is readily proved, which establishes a relationship between best approximations to $f \in C[0, 1]$ from $S_{m,n}^r(\Delta')$ and $K_{m,n}^p$, where $\Delta' = \{\xi_i\}_{i=1}^r$.

THEOREM 4.2. *For* $1 \leqslant p < \infty$, *let* g_p^* *be a best* L_p *approximation to* $f \in C[0, 1]$ *from* $K_{m,n}^p$ *with all assumptions in Theorem* 4.1 *satisfied. Let* $\Delta' = \{\xi_i\}_{i=1}^r$ *be the distinct zeros of* $H_{p,n}$ *in* $(0, 1)$. *Then*

(i) g_p^* *is the best approximation to* f *from* $S_{m,n}^r(\Delta')$;

(ii) *g_p^* is the best L_p approximation to f from the subspace*

$$S_n^{r***}(\Delta', \phi_p) = \{s \in S_n^r(\Delta') : s^{(i)}(1) = 0, i \in I(\phi_p)\};$$

(iii) *g_1^* is the unique best L_1 approximation to f from $K_{m,n}^p$.*

Proof. (i) It follows directly from Theorem 2.1, 3.3 and 4.1 that g_p^* is the best L_p approximation to f from $S_{m,n}^r(\Delta')$.

(ii) By (i) and Theorem 2.2, g_p^* is the best L_p approximation to f from $S_n^{r*}(\Delta', \phi_p)$. Note $\{\xi_i\}_{i=1}^r$ are the distinct zeros of $H_{p,n}$ in (0, 1). $J(\phi_p)$ is an empty set. Hence, (ii) follows.

(iii) The uniqueness follows from (i) and the fact that there is a unique best L_1 approximation to $f \in C[0, 1]$ from $S_{m,n}^r(\Delta')$ (see [9]).

THEOREM 4.3. *For $1 \leq p < \infty$, let $s_p^* \in S_{m,n}^k(\Delta)$ be the best L_p approximation to $f \in L_p[0, 1]$ from $S_{m,n}^k(\Delta)$. Assume that each knot x_j in Δ is a nontrivial knot of s_p^*. Then s_p^* is a best L_p approximation to f from $K_{m,n}^p$ if and only if*

$$(-1)^m \int_{x_j}^{x} (x-t)^{n-1} \phi(t)\, dt \leq 0, \qquad x \in [x_j, x_{j+1}], \quad j = 0, 1, ..., n. \tag{4.2}$$

Proof. Let s_p^* be a best L_p approximation to f from $K_{m,n}^p$. By Theorem 3.3, we have $(-1)^m H_{p,n}(x) \leq 0$ for $x \in [0, 1]$. Since x_j is a nontrivial knot of s_p^*, $(-1)^m H_{p,n}(x_j) = 0$ for $j \in N_k$. Hence,

$$\begin{aligned}(-1)^m \int_{x_j}^{x} (x-t)^{n-1} \phi(t)\, dt &= (-1)^m H_{m,n}(x) - (-1)^m H_{m,n}(x_j) \\ &= (-1)^m H_{m,m}(x),\end{aligned} \tag{4.3}$$

and thus, (4.2) holds.

Conversely, let (4.2) hold. By (4.3), we have $(-1)^m H_{p,n}(x) \leq 0$ for $x \in [0, 1]$. Conditions (i), (ii), (iv), and (v) of Theorem 2.1, and the above inequality imply conditions (i)–(v) of Theorem 3.3. Hence, s_p^* is a best L_p approximation to f from $K_{m,n}^p$. This completes the proof.

As an application, let us establish an interesting relationship between best n-convex L_p approximation and best L_p approximation by n-convex splines of degree $n-1$. Let $K_{n,p}$ denote the set of n-convex functions in $L_p[0, 1]$.

COROLLARY 4.1. *Let $f \in C[0, 1]$. For $1 \leq p < \infty$, let $g_p^* \in K_{n,p}$ such that $f \neq g_p^*$ a.e. in $[0, 1]$ and $f - g_p^*$ has a finite number of sign changes in $(0, 1)$. Define ϕ_p as (4.1) and $H_{p,n}$ as before. If g_p^* is a best L_p approximation to* *f from $K_{n,p}$, then g_p^* is a best L_p approximation to f from $S'_{n,n}(\Delta')$, where $\Delta': 0<\xi_1<\cdots<\xi_r<1$, and the ξ_i's are the distinct zeros of $H_{p,n}$ in* $(0, 1)$.

5. Best Monotone Convex L_p Approximation

As applications of the results in Section 3, we consider best L_p approximation by monotone convex functions, and the relationship between best convex L_p approximation and best monotone convex L_p approximation.

For $1 \leqslant p < \infty$, let $M_D(a, b)$ (resp., $M_I(a, b)) \subset L_p[a, b]$ be the set of nonincreasing (resp., nondecreasing) convex functions on (a, b). Thus, $g(x) \in M_D(a, b)$ if and only if $G(x) \equiv g(-x) \in M_I(-b, -a)$. In addition, $g^*(x)$ is a best L_p approximation to f from $M_D(a, b)$ if and only if $G^*(x) \equiv g^*(-x)$ is a best L_p approximation to $F(x) = f(-x)$ from $M_I(-b, -a)$.

Since a nondecreasing convex function is $(1.2)_\sigma$-convex with $\sigma = (1, -1)$ and a nonincreasing function is $(1, 2)_\sigma$-convex with $\sigma = (-1, -1)$, a similar reasoning to the proof of Theorem 3.3 gives the following two corollaries of Theorem 3.3:

Corollary 5.1. (a) *For $1 < p < \infty$, $g^* \in M_D(a, b)$ (resp., $M_I(a, b)$) is the best nonincreasing (resp., nondecreasing) convex L_p approximation to $f \in L_p[a, b]$ if and only if*

(i) $\int_a^b \phi_p(x)\,dx = 0$;

(ii) $\int_a^t (t-x)\,\phi_p(x)\,dx \leqslant 0$ *(resp., $\int_t^b (x-t)\,\phi_p(x)\,dx \leqslant 0$) for all* $t \in [a, b]$;

(iii) *if $\int_a^b x\phi_p(x)\,dx > 0$ (resp., $\int_a^b x\phi_p(x)\,dx < 0$), then $g_p^{*\prime}(b\ \ ) = 0$ (resp., $g_{p-}^{*\prime}(a^+) = 0$)*;

(iv) *if $\int_a^{t_0} (t_0 - x)\,dx < 0$ (resp., $\int_{t_0}^b (x - t_0)\,\phi_p(x)\,dx < 0$) for some $t_0 \in (a, b)$, then g_p^* is a linear polynomial in a neighborhood of t_0.*

(b) *For $p = 1$, $g_1^* \in M_D(a, b)$ (resp., $M_I(a, b)$) is a best nonincreasing (resp., nondecreasing) convex L_1 approximation to $f \in L_1[a, b]$ if and only if there exists a $\phi_1 \in L_\infty[a, b]$ with $\|\phi_1\|_\infty = 1$, $\int_a^b \phi_1(f - g_1^*) = \|f - g_1^*\|_1$ satisfying conditions* (i)–(iv) *in* (a) *with* $p = 1$.

The next three theorems establish some relationships between best convex L_p approximation and bes monotone convex L_p approximation.

Theorem. 5.1. *Let g_p^* be a best convex L_p approximation to $f \in L_p[0, 1]$, for $1 \leqslant p < \infty$. Then, there exists a $t \in [0, 1]$ such that g_p^* is* *both a best nonincreasing convex L_p approximation to f on $[0, t]$ and a best nondecreasing convex L_p approximation to f on $[t, 1]$.*

Proof. If g_p^* is nonincreasing (nondecreasing) on $(0, 1)$, then let $t = 1$ $(t = 0)$. Assume that g_p^* is a nonmonotone convex function. Let

$$m = \inf\{g_p^*(x): x \in [0, 1]\}.$$

Then the set $A = \{x \in [0, 1]: g_p^*(x) = m\}$ is a nonempty and closed interval contained in $(0, 1)$. Define $t = \inf A$. Then, g_p^* is nonincreasing on $(0, t)$ and nondecreasing on $(t, 1)$. By the definition of t, g_p^* cannot be a linear polynomial in any neighborhood of t which contains t as an interior point. The characterization of best convex approximation implies $\int_0^t (t-x)\,\phi_p(x)\,dx = 0$. Thus, g_p^* is a best approximation to f on both $[0, t]$ and $[t, 1]$ (see [15, 19]). Since the set of nonincreasing convex functions in $L_p[0, t]$ is contained in the set of convex functions in $L_p[0, t]$, g_p^* is also a best nonincreasing convex approximation to f on $[0, t]$. Similarly, g_p^* is a best nondecreasing convex approximation to f on $[t, 1[$.

THEOREM 5.2. *For $1 < p < \infty$ let $f \in L_p[0, 1]$. Assume $t \in (0, 1)$. Let $g_D \in M_D(0, t)$ (resp., $g_I \in M_I(t, 1)$) be the best nonincreasing (resp., nondecreasing) convex L_p approximation to f on $[0, t]$ (resp., on $[t, 1]$). Define*

$$\phi_{p,D}(x) = \operatorname{sign}[f(x) - g_D(x)]\,|f(x) - g_D(x)|^{p-1}, \qquad \text{for } x \in [0, t],$$

$$\phi_{p,I}(x) = \operatorname{sign}[f(x) - g_I(x)]\,|f(x) - g_I(x)|^{p-1}, \qquad \text{for } x \in [t, 1],$$

and

$$g(x) = \begin{cases} g_D(x), & x \in [0, t] \\ g_I(x), & x \in (t, 1]. \end{cases}$$

Then, g is the best convex L_p approximation to f on $[0, 1]$ if and only if

(i) $g_D(t) = g_I(t)$,

(ii) $\int_0^t (t-x)\,\phi_{p,D}(x)\,dx = \int_t^1 (x-t)\,\phi_{p,I}(x)\,dx$.

Proof. Let

$$\phi_p(x) = \begin{cases} \phi_{p,D}(x), & x \in [0, t] \\ \phi_{p,I}(x), & x \in (t, 1]. \end{cases}$$

Assume g is the best convex L_p approximation to f on $[0, 1]$. Then g is continuous on $(0, 1)$ and thus $g_D(t) = g_I(t)$. In addition, by the characterization of best convex L_p approximation, we have $\int_0^1 \phi_p = 0$, and $\int_0^1 x\phi_p(x)\,dx = 0$. Hence,

$$\int_0^1 (t-x)\,\phi_p(x)\,dx = 0 \qquad \text{for all} \quad t \in (0, 1).$$

It follows from this equation that (ii) holds.

Condition (i) with the facts that g_{D} is nonincreasing convex on $[0, t]$ and g_{I} is nondecreasing convex on $[t, 1]$ implies that g is convex on $[0, 1]$. By the assumptions, we find

$$\int_0^1 \phi_p(x)\,dx = \int_0^t \phi_{p,\mathrm{D}}(x)\,dx + \int_t^1 \phi_{p,\mathrm{I}}(x)\,dx = 0$$

and

$$\int_0^1 x\phi_p(x)\,dx = \int_0^t (t-x)\,\phi_{p,\mathrm{D}}(x)\,dx + \int_t^1 (t-x)\,\phi_{p,\mathrm{I}}(x)\,dx = 0.$$

For $x \in [0, t]$,

$$\int_0^x (x-u)\,\phi_p(u)\,du = \int_0^x (x-u)\,\phi_{p,\mathrm{D}}(u)\,du \leqslant 0,$$

and for $x \in (t, 1]$, by condition (ii),

$$\begin{aligned}
&\int_0^x (x-u)\,\phi_p(u)\,du \\
&\quad = \int_0^t (x-u)\,\phi_{p,\mathrm{D}}(u)\,du + \int_t^x (x-u)\,\phi_{p,\mathrm{I}}(u)\,du \\
&\quad = \int_0^t (t-u)\,\phi_{p,\mathrm{D}}(u)\,du + \int_t^x (x-t)\,\phi_{p,\mathrm{I}}(u)\,du + \int_t^x (t-u)\,\phi_{p,\mathrm{I}}(u)\,du \\
&\quad = \int_t^1 (u-t)\,\phi_{p,\mathrm{I}}(u)\,du - \int_t^x (u-t)\,\phi_{p,\mathrm{I}}(u)\,du + \int_t^x (x-t)\,\phi_{p,\mathrm{I}}(u)\,du \\
&\quad = \int_x^1 (u-t)\,\phi_{p,\mathrm{I}}(u)\,du - \int_x^1 (x-t)\,\phi_{p,\mathrm{I}}(u)\,du \\
&\quad = \int_x^1 (u-x)\,\phi_{p,\mathrm{I}}(u)\,du \leqslant 0.
\end{aligned}$$

Assume that for some $x_0 \in (0, 1)$, $\int_0^{x_0} (x_0-u)\,\phi_p(u)\,du < 0$. If $x_0 \in (0, t)$, then g_{D} is a linear polynomial in a neighborhood of x_0 and so is g. If $x_0 \in (t, 1)$, then by the above reasoning, we have $\int_{x_0}^1 (u-x_0)\,\phi_{p,\mathrm{I}}(u)\,du<0$. Thus, g_{I} is a linear polynomial in a neighborhood of x_0 and so is g. If $x_0=t$, in view of the continuity of $\int_0^x (x-u)\,\phi_p(u)\,du$ for $x\in[0,1]$,

$$\int_0^x (x-u)\,\phi_p(u)\,du<0, \qquad x\in(t-\delta_1, t], \text{ for some } \delta_1>0.$$

By the characterization of best nonincreasing convex L_p approximation, we find that $g'_-(t^-)=g'_{\mathrm{D}-}(t^-)=0$ and g is a linear polynomial on $(t-\delta_1, t]$. In addition, since (ii) holds, $\int_t^1 (x-t)\,\phi_{p,\mathrm{I}}(x)\,dx<0$. Similarly, $g'_+(t^+)=g'_{I+}(t^+)=0$, and g is a linear polynomial on $[t, t+\delta_2)$ for some $\delta_2>0$. Hence,

$$0=g'_-(t^-)\leqslant g'(t)\leqslant g'_+(t^+)=0.$$

Thus $g'(t)$ exists and vanishes. Therefore g is a constant on $(t-\delta_1, t+\delta_2)$.

The conditions that we verify guarantee that g is the best convex L_p approximation to f on $[0, 1]$.

For $p=1$, we have the following similar result:

THEOREM 5.3. *Let $f\in L_1[0,1]$ and $t\in(0,1)$. Assume $g_{\mathrm{D}}\in M_{\mathrm{D}}(0,t)$ (resp., $g_{\mathrm{I}}\in M_{\mathrm{I}}(t,1)$) is a best nonincreasing (resp., nondecreasing) convex L_1 approximation to f on $[0,t]$ (resp., on $[t,1]$). Define*

$$g(x)=\begin{cases} g_{\mathrm{D}}(x) & x\in[0,t]\\ g_{\mathrm{I}}(x) & x\in(t,1],\end{cases}$$

Let $\Phi(g_{\mathrm{D}})$ be the set of $\phi\in L_\infty[0,t]$ with $\|\phi\|_\infty=1$ and $\int_0^t \phi(f-g_{\mathrm{D}})=\|f-g_{\mathrm{D}}\|_1$, satisfying conditions (i)–(v) of Corollary 5.1. Let $\Phi(g_{\mathrm{I}})$ be the set of $\phi\in L_\infty[t,1]$ with $\|\phi\|_\infty=1$ and $\int_t^1 \phi(f-g_{\mathrm{I}})=\|f-g_{\mathrm{I}}\|_1$, satisfying conditions (i)–(v) of Corollary 5.1. Then, g is a best convex L_1 approximation to f on $[0,1]$ if and only if

(i) $g_{\mathrm{D}}(t)=g_{\mathrm{I}}(t)$,

(ii) *there exist $\phi_D\in\Phi(g_{\mathrm{D}})$ and $\phi_{\mathrm{I}}\in\Phi(g_{\mathrm{I}})$ such that*

$$\int_0^t (t-x)\,\phi_{\mathrm{D}}(x)\,dx=\int_t^1 (x-t)\,\phi_{\mathrm{I}}(x)\,dx.$$

Proof. Let

$$\phi(x)=\begin{cases} \phi_{\mathrm{D}}(x) & x\in[0,t]\\ \phi_{\mathrm{I}}(x) & x\in(t,1].\end{cases}$$

Then, $\|\phi\|_\infty = 1$ and

$$\int_0^1 \phi(f-g) = \int_0^t \phi_D(f-g_D) + \int_t^1 \phi_I(f-g_I) = \|f-g\|_1.$$

The rest of this proof is similar to the proof of Theorem 5.2. This completes the proof of Theorem 5.3.

Remark. All results in this paper could be generalized to Tchebycheffian splines and to functions generalized convex with respect to an ECT-system.

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